2次元不等分割中心差分によるナビエ-ストーク方程式

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1. 離散化対象式（保存形を使用）

下記の速度-圧力法でのナビエ-ストーク方程式を通常格子で離散化する。

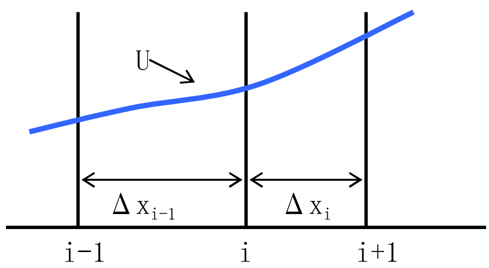
$$-\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 p}{\partial y^2}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)/\Delta t$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial (uv)}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+\frac{\partial (uv)}{\partial x}+\frac{\partial v^2}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

2. 不等分割計算の方法

不等分割中心差分は下記に1次元の例で示す様に、線形補間して求める。



$$\frac{\partial u}{\partial x}=\frac{u_{i+1}-u_i}{2\Delta x_i}+\frac{u_i-u_{i-1}}{2\Delta x_{i-1}}$$

$$\frac{\partial^2 u}{\partial x^2}=\left(\frac{u_{i+1}-u_i}{\Delta x_i}\frac{\Delta x}{\Delta x_i}-\frac{u_i-u_{i-1}}{\Delta x_{i-1}}\frac{\Delta x}{\Delta x_{i-1}}\right)/\Delta x=\frac{u_{i+1}-u_i}{(\Delta x_i)^2}-\frac{u_i-u_{i-1}}{(\Delta x_{i-1})^2}$$

3. ナビエストークス方程式の離散化

3.1　Pの離散化

$$\frac{u_{i,j}-u_{i+1,j}}{(\Delta x_i)^2}+\frac{u_{i,j}-u_{i-1,j}}{(\Delta x_{i-1})^2}+\frac{v_{i,j}-v_{i,j+1}}{(\Delta y_j)^2}+\frac{v_{i,j}-v_{i,j-1}}{(\Delta y_{j-1})^2}$$

$$=\left(\frac{u_{i+1,j}-u_{i,j}}{2\Delta x_i}+\frac{u_{i,j}-u_{i-1,j}}{2\Delta x_{i-1}}\right)^2+\left(\frac{v_{i,j+1}-v_{i,j}}{2\Delta y_j}+\frac{v_{i,j}-v_{i,j-1}}{2\Delta y_{j-1}}\right)^2$$

$$+2\left(\frac{v_{i+1,j}-v_{i,j}}{2\Delta x_i}+\frac{v_{i,j}-v_{i-1,j}}{2\Delta x_{i-1}}\right)\left(\frac{u_{i,j+1}-u_{i,j}}{2\Delta y_j}+\frac{u_{i,j}-u_{i,j-1}}{2\Delta y_{j-1}}\right)$$

$$-\left(\frac{u_{i+1,j}-u_{i,j}}{2\Delta x_i}+\frac{u_{i,j}-u_{i-1,j}}{2\Delta x_{i-1}}+\frac{v_{i,j+1}-v_{i,j}}{2\Delta y_j}+\frac{v_{i,j}-v_{i,j-1}}{2\Delta y_{j-1}}\right)/\mathrm{Re}$$

3.2　U の離散化

$$\frac{u_{i,j}^{(k+1)}-u_{i,j}}{\Delta t}=-\left(\frac{u_{i+1,j}^2-u_{i,j}^2}{2\Delta x_i}+\frac{u_{i,j}^2-u_{i-1,j}^2}{2\Delta x_{i-1}}\right)-\left(\frac{u_{i,j+1}v_{i,j+1}-u_{i,j}v_{i,j}}{2\Delta y_j}+\frac{u_{i,j}v_{i,j}-u_{i,j-1}v_{i,j-1}}{2\Delta y_{j-1}}\right)$$

$$-\left(\frac{p_{i+1,j}-p_{i,j}}{2\Delta x_i}+\frac{p_{i,j}-p_{i-1,j}}{2\Delta x_{i-1}}\right)$$

$$-\left(\frac{u_{i,j}-u_{i+1,j}}{(\Delta x_i)^2}+\frac{u_{i,j}-u_{i-1,j}}{(\Delta x_{i-1})^2}+\frac{u_{i,j}-u_{i,j+1}}{(\Delta y_i)^2}+\frac{u_{i,j}-u_{i,j-1}}{(\Delta y_{i-1})^2}\right)$$

3.3　V の離散化

$$\frac{v_{i,j}^{(k+1)}-v_{i,j}}{\Delta t}=-\left(\frac{v_{i,j+1}^2-v_{i,j}^2}{2\Delta y_i}+\frac{v_{i,j}^2-v_{i,j-1}^2}{2\Delta y_{i-1}}\right)-\left(\frac{u_{i+1,j}v_{i+1,j}-u_{i,j}v_{i,j}}{2\Delta x_j}+\frac{u_{i,j}v_{i,j}-u_{i-1,j}v_{i-1,j}}{2\Delta x_{j-1}}\right)$$

$$-\left(\frac{p_{i,j+1}-p_{i,j}}{2\Delta y_j}+\frac{p_{i,j}-p_{i,j-1}}{2\Delta y_{j-1}}\right)$$

$$-\left(\frac{v_{i,j}-v_{i+1,j}}{(\Delta x_i)^2}+\frac{v_{i,j}-v_{i-1,j}}{(\Delta x_{i-1})^2}+\frac{v_{i,j}-v_{i,j+1}}{(\Delta y_i)^2}+\frac{v_{i,j}-v_{i,j-1}}{(\Delta y_{i-1})^2}\right)$$