3 次元中心差分によるナビエ-ストーク方程式

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1.　基礎方程式

$$div(v)=0$$

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\frac{1}{\mathrm{Re}}\Delta v$$

2.　離散化対象式

下記の速度-圧力法でのナビエ-ストーク方程式を通常格子で離散化する。

（1）div,∇ での表示

$$-\Delta p=div(v\cdot\nabla)v-div(v)/\Delta t$$

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\frac{1}{\mathrm{Re}}\Delta v$$

（2）$div(v\cdot\nabla)v,(v\cdot\nabla)v$ の x,y,z 微分による表示

$$D=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$div(v\cdot\nabla)v=\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\frac{\partial}{\partial z}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial w}{\partial x}\cdot\frac{\partial u}{\partial z}+\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial z}+\left(\frac{\partial w}{\partial z}\right)^2+\frac{\partial w}{\partial x}\cdot\frac{\partial u}{\partial z}$$

$$+\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial z}+\frac{\partial D}{\partial x}+\frac{\partial D}{\partial y}+\frac{\partial D}{\partial y}$$

$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+2\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}+2\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial z}+2\frac{\partial u}{\partial z}\cdot\frac{\partial w}{\partial x}$$

$$(v\cdot\nabla)u=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{\partial}{\partial x}\left(u^2\right)+\frac{\partial}{\partial y}(uv)+\frac{\partial}{\partial z}(uw)+u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}\left(u^2\right)+\frac{\partial}{\partial y}(uv)+\frac{\partial}{\partial z}(uw)$$

$$(v\cdot\nabla)v=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{\partial}{\partial x}(uv)+\frac{\partial}{\partial y}\left(v^2\right)+\frac{\partial}{\partial z}(vw)$$

$$(v\cdot\nabla)w=u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{\partial}{\partial x}(uw)+\frac{\partial}{\partial y}(vw)+\frac{\partial}{\partial z}\left(w^2\right)$$

（3）保存形での x, y, z 微分表示

$$-\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 p}{\partial y^2}-\frac{\partial^2 p}{\partial z^2}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+2\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}\right)$$
$$-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)/\Delta t$$

$$\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+\frac{\partial (uv)}{\partial x}+\frac{\partial (v^2)}{\partial y}+\frac{\partial (vw)}{\partial z}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+\frac{\partial (uw)}{\partial x}+\frac{\partial (vw)}{\partial y}+\frac{\partial (w^2)}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

（4）非保存形での x, y, z 微分表示

$$-\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 p}{\partial y^2}-\frac{\partial^2 p}{\partial z^2}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+2\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}\right)$$
$$-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)/\Delta t$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

3.　中心差分での離散化計算式

3.1　P の離散化

保存形、非保存形共に下記を使用する。

$$\frac{2p_{i,j,k}-p_{i-1,j,k}-p_{i+1,j,k}}{(\Delta x)^2}+\frac{2p_{i,j,k}-p_{i,j-1,k}-p_{i,j+1,k}}{(\Delta y)^2}+\frac{2p_{i,j,k}-p_{i,j-1,k}-p_{i,j+1,k}}{(\Delta z)^2}=$$

$$\frac{(u_{i+1,j,k}-u_{i-1,j,k})^2}{4(\Delta x)^2}+\frac{(v_{i,j+1,k}-v_{i,j-1,k})^2}{4(\Delta y)^2}+\frac{(v_{i,j,k+1}-v_{i,j,k-1})^2}{4(\Delta z)^2}$$

$$+\frac{(v_{i+1,j,k}-v_{i-1,j,k})(u_{i,j+1,k}-u_{i,j-1,k})}{2\Delta x\Delta y}+\frac{(w_{i,j+1,k}-w_{i,j-1,k})(v_{i,j,k+1}-v_{i,j,k-1})}{2\Delta y\Delta z}$$

$$+\frac{(u_{i,j,k+1}-u_{i,j,k-1})(w_{i+1,j,k}-w_{i-1,j,k})}{2\Delta z\Delta x}$$

$$-\left(\frac{u_{i+1,j,k}-u_{i-1,j,k}}{2\Delta x}+\frac{v_{i,j+1,k}-v_{i,j-1,k}}{2\Delta y}+\frac{w_{i,j,k+1}-w_{i,j,k-1}}{2\Delta z}\right)/\Delta t$$

3.2 保存形の離散化

時刻$(n+1)\Delta t$の値は$u_{i,j,k}^{(n+1)}$のように右肩に$(n+1)$のインデックスを付けて表示

時刻$n\Delta t$の値は$u_{i,j,k}$のように右肩の(n)のインデックスは省いて表示する。

(1) uの離散化

$$\frac{u_{i,j,k}^{(n+1)}-u_{i,j,k}}{\Delta t}=-\frac{u_{i+1,j,k}^2-u_{i-1,j,k}^2}{2\Delta x}-\frac{u_{i,j+1,k}v_{i,j+1,k}-u_{i,j-1,k}v_{i,j-1,k}}{2\Delta y}$$

$$-\frac{u_{i,j,k+1}w_{i,j,k+1}-u_{i,j,k+1}w_{i,j,k-1}}{2\Delta z}-\frac{p_{i+1,j,k}-p_{i-1,j,k}}{2\Delta x}$$

$$+\left(\frac{u_{i+1,j,k}-2u_{i,j,k}+u_{i-1,j,k}}{(\Delta x)^2}+\frac{u_{i,j+1,k}-2u_{i,j,k}+u_{i,j+1,k}}{(\Delta y)^2}+\frac{u_{i,j,k+1}-2u_{i,j,k}+u_{i,j,k-1}}{(\Delta z)^2}\right)/\mathrm{Re}$$

(2) vの離散化

$$\frac{v_{i,j,k}^{(n+1)}-v_{i,j,k}}{\Delta t}=-\frac{u_{i+1,j,k}v_{i+1,j,k}-u_{i-1,j,k}v_{i-1,j,k}}{2\Delta x}-\frac{v_{i,j+1,k}^2-v_{i,j-1,k}^2}{2\Delta y}$$

$$-\frac{v_{i,j,k+1}w_{i,j,k+1}-v_{i,j,k+1}w_{i,j,k-1}}{2\Delta z}-\frac{p_{i,j+1,k}-p_{i,j-1,k}}{2\Delta y}$$

$$+\left(\frac{v_{i+1,j,k}-2v_{i,j,k}+v_{i-1,j,k}}{(\Delta x)^2}+\frac{v_{i,j+1,k}-2v_{i,j,k}+v_{i,j-1,k}}{(\Delta y)^2}+\frac{v_{i,j,k+1}-2v_{i,j,k}+v_{i,j,k-1}}{(\Delta z)^2}\right)/\mathrm{Re}$$

（3） w の離散化

$$\frac{w_{i,j,k}^{(n+1)} - w_{i,j,k}}{\Delta t} = -\frac{u_{i+1,j,k} w_{i+1,j,k} - u_{i-1,j,k,} w_{i-1,j,k}}{2\Delta x} - \frac{v_{i,j+1,k} w_{i,j+1,k} - v_{i,j-1,k} w_{i,j-1,k}}{2\Delta y}$$

$$- \frac{w_{i,j,k+1}^2 - v_{i,j,k-1}^2}{2\Delta z} - \frac{p_{i,j,k+1} - p_{i,j,k-1}}{2\Delta z}$$

$$+ \left(\frac{v_{i+1,j,k} - 2v_{i,j,k} + v_{i-1,j,k}}{(\Delta x)^2} + \frac{v_{i,j+1,k} - 2v_{i,j,k} + v_{i,j-1,k}}{(\Delta y)^2} + \frac{v_{i,j,k+1} - 2v_{i,j,k} + v_{i,j,k-1}}{(\Delta z)^2} \right) / \mathrm{Re}$$

3.3 非保存形の離散化

（1） u の離散化

$$\frac{u_{i,j,k}^{(n+1)} - u_{i,j,k}}{\Delta t} = -u_{i,j,k} \cdot \frac{u_{i+1,j,k} - u_{i-1,j,k}}{2\Delta x} - v_{i,j,k} \cdot \frac{u_{i,j+1,k} - u_{i,j-1,k}}{2\Delta y}$$

$$- w_{i,j,k} \cdot \frac{u_{i,j,k+1} - u_{i,j,k+1}}{2\Delta z} - \frac{p_{i+1,j,k} - p_{i-1,j,k}}{2\Delta x}$$

$$+ \left(\frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta y)^2} + \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\Delta z)^2} \right) / \mathrm{Re}$$

（2） v の離散化

$$\frac{v_{i,j,k}^{(n+1)} - v_{i,j,k}}{\Delta t} = -u_{i,j,k} \cdot \frac{v_{i+1,j,k} - v_{i-1,j,k}}{2\Delta x} - v_{i,j,k} \cdot \frac{v_{i,j+1,k} - v_{i,j-1,k}}{2\Delta y}$$

$$- w_{i,j,k} \cdot \frac{v_{i,j,k+1} - v_{i,j,k+1}}{2\Delta z} - \frac{p_{i,j+1,k} - p_{i,j-1,k}}{2\Delta y}$$

$$+ \left(\frac{v_{i+1,j,k} - 2v_{i,j,k} + v_{i-1,j,k}}{(\Delta x)^2} + \frac{v_{i,j+1,k} - 2v_{i,j,k} + v_{i,j-1,k}}{(\Delta y)^2} + \frac{v_{i,j,k+1} - 2v_{i,j,k} + v_{i,j,k-1}}{(\Delta z)^2} \right) / \mathrm{Re}$$

（3） w の離散化

$$\frac{w_{i,j,k}^{(n+1)} - w_{i,j,k}}{\Delta t} = -u_{i,j,k} \cdot \frac{w_{i+1,j,k} - w_{i-1,j,k}}{2\Delta x} - v_{i,j,k} \cdot \frac{w_{i,j+1,k} - w_{i,j-1,k}}{2\Delta y}$$

$$- w_{i,j,k} \cdot \frac{w_{i,j,k+1} - w_{i,j,k+1}}{2\Delta z} - \frac{p_{i,j,k+1} - p_{i,j,k-1}}{2\Delta z}$$

$$+ \left(\frac{w_{i+1,j,k} - 2w_{i,j,k} + w_{i-1,j,k}}{(\Delta x)^2} + \frac{w_{i,j+1,k} - 2w_{i,j,k} + w_{i,j-1,k}}{(\Delta y)^2} + \frac{w_{i,j,k+1} - 2w_{i,j,k} + w_{i,j,k-1}}{(\Delta z)^2} \right) / \mathrm{Re}$$