2次元中心差分によるナビエ-ストーク方程式

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1.　基礎方程式

$$div(v)=0$$

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\frac{1}{\mathrm{Re}}\Delta v$$

2.　離散化対象式

下記の速度-圧力法でのナビエ-ストーク方程式を通常格子で離散化する。

(1) div,∇ での表示

$$-\Delta p=div(v\cdot\nabla)v-div(v)/\Delta t$$

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v=-\nabla p+\frac{1}{\mathrm{Re}}\Delta v$$

(2) $div(v\cdot\nabla)v,(v\cdot\nabla)v$ の x,y,z 微分による表示

$$D=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\begin{aligned}div(v\cdot\nabla)v&=\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)\\&=\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}+\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial D}{\partial x}+\frac{\partial D}{\partial y}\\&=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\end{aligned}$$

$$(v\cdot\nabla)u=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial x}\left(u^2\right)+\frac{\partial}{\partial y}(uv)+u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial x}\left(u^2\right)+\frac{\partial}{\partial y}(uv)$$

$$(v\cdot\nabla)v=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}(uv)+\frac{\partial}{\partial y}\left(v^2\right)$$

(3) 保存形での x,y,z 微分表示

$$-\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 p}{\partial y^2}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)/\Delta t$$

$$\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial x}+\frac{\partial (uv)}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+\frac{\partial (uv)}{\partial x}+\frac{\partial (v^2)}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

（4）非保存形での x, y, z 微分表示

$$-\frac{\partial^2 p}{\partial x^2}-\frac{\partial^2 p}{\partial y^2}=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)/\Delta t$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

3.　中心差分での離散化計算式

3.1　Ｐの離散化

保存形、非保存形共に下記を使用する。

$$\frac{2p_{i,j}-p_{i-1,j}-p_{i+1,j}}{(\Delta x)^2}+\frac{2p_{i,j}-p_{i,j-1}-p_{i,j+1}}{(\Delta y)^2}=\frac{(u_{i+1,j}-u_{i-1,j})^2}{4(\Delta x)^2}+\frac{(v_{i,j+1}-v_{i,j-1})^2}{4(\Delta y)^2}$$
$$+\frac{(v_{i+1,j}-v_{i-1,j})(u_{i,j+1}-u_{i,j-1})}{2\Delta x\Delta y}-\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\right)/\Delta t$$

3.2 保存形の離散化

時刻 $(n+1)\Delta t$ の値は $u_{i,j,k}^{(n+1)}$ のように右肩に $(n+1)$ のインデックスを付けて表示

時刻 $n\Delta t$ の値は $u_{i,j,k}$ のように右肩の (n) のインデックスは省いて表示する。

（1） u の離散化

$$\frac{u_{i,j}^{(n+1)} - u_{i,j}}{\Delta t} = -\frac{u_{i+1,j}^2 - u_{i-1,j}^2}{2\Delta x} - \frac{u_{i,j+1}v_{i,j+1} - u_{i,j-1}v_{i,j-1}}{2\Delta y} - \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x}$$
$$+\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2}\right) / \mathrm{Re}$$

（2） v の離散化

$$\frac{v_{i,j}^{(n+1)} - v_{i,j}}{\Delta t} = -\frac{u_{i+1,j}v_{i+1,j} - u_{i-1,j}v_{i-1,j}}{2\Delta x} - \frac{v_{i,j+1}^2 - v_{i,j-1}^2}{2\Delta y} - \frac{p_{i,j+1} - p_{i,j-1}}{2\Delta y}$$
$$+\left(\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(\Delta x)^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{(\Delta y)^2}\right) / \mathrm{Re}$$

3.3 非保存形の離散化

（1） u の離散化

$$\frac{u_{i,j}^{(n+1)} - u_{i,j}}{\Delta t} = -u_{i,j} \cdot \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} - v_{i,j} \cdot \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} - \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x}$$
$$+\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2}\right) / \mathrm{Re}$$

（2） v の離散化

$$\frac{v_{i,j}^{(n+1)} - v_{i,j}}{\Delta t} = -u_{i,j} \cdot \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - v_{i,j} \cdot \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} - \frac{p_{i,j+1} - p_{i,j-1}}{2\Delta y}$$
$$+\left(\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(\Delta x)^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{(\Delta y)^2}\right) / \mathrm{Re}$$